# The Hodrick-Prescott filter with priors: linear restrictions on HP filters

Juan Manuel Julio

Banco de la Republica, Universidad Nacional de Colombia

19. October 2011

# The Hodrick-Prescott Filter with Priors: Linear Restrictions on HP Filters[†]

Juan Manuel Julio[‡]

**Abstract**

A closed formula for the Hodrick & Prescott, HP, filter subject to linear restrictions is derived. This filter is also known as the HP filter with priors. When the formula is applied to the ordinary HP filter linear restrictions apply only within the sample. However, when this formula is applied to the extended HP filter and extensions that correct for GDP revisions and delays, linear restrictions apply out of sample also.

*JEL:E32, C22*

*Key Words: Business Cycles, Hodrick-Prescott Filter*

[†] The conclusions and opinions contained in this note, as well any remaining errors are the sole responsibility of its author and do not compromise the opinions of BANCO DE LA REPUBLICA, its Board of Governors on Universidad Nacional de Colombia.

[‡] jjulioro@banrep.gov.co. The author is a senior researcher working for the Research Unit of BANCO DE LA REPUBLICA and is also an associate professor at the Department of Statistic of the Universidad Nacional de Colombia.

## 1. Introduction

In spite of the fact that the Hodrick & Prescott filter has been heavily criticized and might not be optimal under reasonable macoeconomic circumstances, its application is pervasive in macro-econometric practice. See King & Rebelo (1993) and Cogley & Nason (1995), for instance.

In order to improve on the well-known drawbacks of the HP filter, it has been suggested to include external information, drawn from experts, in the form of linear restrictions on the trend values. In this note a closed formula is derived for this purpose. This formula extends to three members of the HP filter family, the ordinary HP filter, the extended HP filter (Kaiser & Maravall (1997) and (1999)), and the further extension that corrects for GDP data delays and revisions (Julio (2011a) and (2011b)).

Under the ordinary HP filter these restrictions constrain the trend on the sample time span. Under the extended and further extension of the HP filter these restrictions apply out of sample also. The Colombian GDP HP filter decomposition is used as an example.

The note is organized as follows. Section two presents the basic computation of the ordinary HP filter. Section three contains the main result, and sections four to six show applications of the different filters and linear restrictions to Colombian GDP data.

## 2. The Ordinary HP Filter

Let $\{y_t\}_{t=1}^{T}$ be a realization of size $T$ of a discrete time stochastic process $\{Y_t\}_t$ . The problem Hodrick & Prescott (1997), HP, faced was to filter the "trend" $\{\tau_1\}_{t=1}^{T}$ out of this realization under the assumption that $Y_t$ decomposes into the trend $\tau_t$ and cycle $\varsigma_t$ components as

$$Y_t = \tau_t + \varsigma_t$$

for $t = 1, 2, 3, \dots, T$ .

Let $\mathbf{T} = [\tau_t]_{T\times1}$ be the trend vector, $\mathbf{y} = [y_t]_{T\times1}$ be the vector containing the realization of the stochastic process, and $\mathbf{Y} = [Y_t]_{T\times1}$ be the vector containing the stochastic process for $t=1,2,\dots,T$. The ordinary HP filter provides a deterministic solution to the signal extraction problem through the minimization of

$$\min_{\mathbf{T}} \sum_{t=1}^{T} \left( y_t - \tau_t \right)^2 + 2\lambda \sum_{t=2}^{T-1} \left[ \left( \tau_{t+1} - \tau_t \right) - \left( \tau_t - \tau_{t-2} \right) \right]^2 = \min_{\mathbf{T}} L(\mathbf{T})$$

where $\lambda$ is a constant that penalizes the lack of smoothness in the trend component.

In matrix terms the objective function can be written as

$$L = L_1 + L_2 = \left( \mathbf{y} - \mathbf{T} \right)^T \left( \mathbf{Y} - \mathbf{T} \right) + \lambda \mathbf{T}^T \mathbf{C}^T \mathbf{C} \mathbf{T}$$

where

$$\mathbf{C} = \begin{bmatrix} 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -2 & 1 \end{bmatrix}$$

Through vector calculus we obtain $\dfrac{\delta L_1}{\delta \mathbf{T}} = -2\left( \mathbf{y} - \mathbf{T} \right)$ y $\dfrac{\delta L_2}{\delta \mathbf{T}} = 2\lambda \mathbf{C}^T \mathbf{C} \mathbf{T}$, which leads us to $\dfrac{\delta L}{\delta \mathbf{T}} = -2\left( \mathbf{y} - \mathbf{T} \right) + 2\lambda \mathbf{C}^T \mathbf{C} \mathbf{T}$, which provides a convenient formula for computing the HP filter as

$$\hat{\mathbf{T}}_{HP} = \left[ \mathbf{I}_T + \lambda \mathbf{C}^T \mathbf{C} \right]^{-1} \mathbf{y}$$

The original series, in logs, is adjusted for seasonality prior to the application of this formula.

## 3. Linear Restrictions on the Hodrick-Prescott Filter

Let us consider the previous minimization problem subject to a set of $1 \le m \le T$ linear restrictions which may be written as

$$\mathbf{B}_{m \times T} \mathbf{T}_{T \times 1} = \boldsymbol{\tau}_{m \times 1}$$

To solve this problem let $\boldsymbol{\delta}_{m \times 1}$ be a vector containing $m$ Lagrange multipliers, and minimize

$$L = L_1 + L_2 + L_3 = \left( \mathbf{y} - \mathbf{T} \right)^T \left( \mathbf{y} - \mathbf{T} \right) + \lambda \mathbf{T}^T \mathbf{C}^T \mathbf{C} \mathbf{T} + 2\boldsymbol{\delta}^T (\mathbf{B}\mathbf{T} - \boldsymbol{\tau})$$

whose derivatives are $\frac{\delta L}{\delta \mathbf{T}} = -2(\mathbf{y} - \mathbf{T}) + 2\lambda \mathbf{C}^T\mathbf{C}\mathbf{T} + 2\mathbf{B}^T\boldsymbol{\delta}$ and $\frac{\delta L}{\delta \boldsymbol{\delta}} = 2(\mathbf{B}\mathbf{T} - \boldsymbol{\tau})$. From which we obtain the solution $\hat{\mathbf{T}}_{HPR} = \left[\mathbf{I}_T + 2\lambda\mathbf{C}^T\mathbf{C}\right]^{-1}\left(\mathbf{y} + \mathbf{B}^T\boldsymbol{\delta}\right) = \hat{\mathbf{T}}_{HP} + \mathbf{A}\mathbf{B}^T\boldsymbol{\delta}$ where $\mathbf{A} = \left[\mathbf{I}_T + \lambda\mathbf{C}^T\mathbf{C}\right]^{-1}$. By replacing $\mathbf{A}$ in the solution we obtain

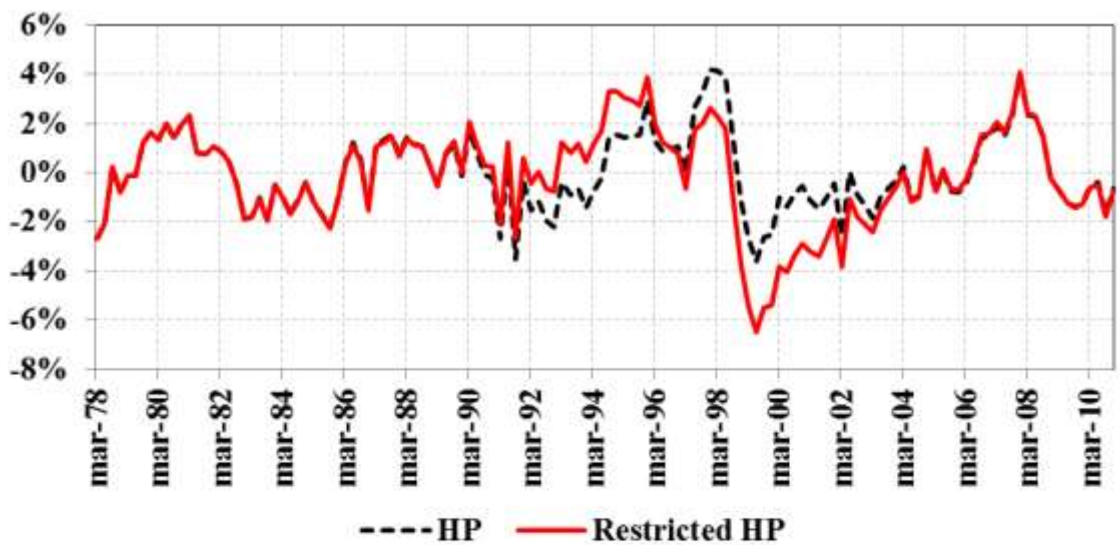
$$\hat{\mathbf{T}}_{HPR} = \hat{\mathbf{T}}_{HP} + \mathbf{A}\mathbf{B}^T\left(\mathbf{B}\mathbf{A}\mathbf{B}^T\right)^{-1}\left(\boldsymbol{\tau} - \mathbf{B}\hat{\mathbf{T}}_{HP}\right)$$

which easily satisfy the restriction.

In the following we will impose four linear restrictions to the estimation of the Colombian output gap. First, the average gap over the four quarters of 1994 is restricted to be 2.38%. Second, the average gap in 1999 is constrained to be -5.72%. Third, the average gap for 2010 is -0.92%. And finally, the average gap for 2011 will be 0.08%. The last restriction constrain out of sample gaps as the dataset runs to 2010Q4.

Figure 1 depicts the ordinary (HPFLPIBSA) and constrained (RESHPFILTPIBSA) output gap from the first three restrictions. The filter is applied to log GDP quarterly figures after adjusting for seasonality. Restrictions imposed on the average gap of 1994 and 1999 have an important effect on the estimated gap, which extends over a long time span. However, the restriction imposed over the values of 2010 does not seem to have an important effect on the gap.

**Figure 1. Ordinary and Restricted Ordinary HP Output Gap Estimates**



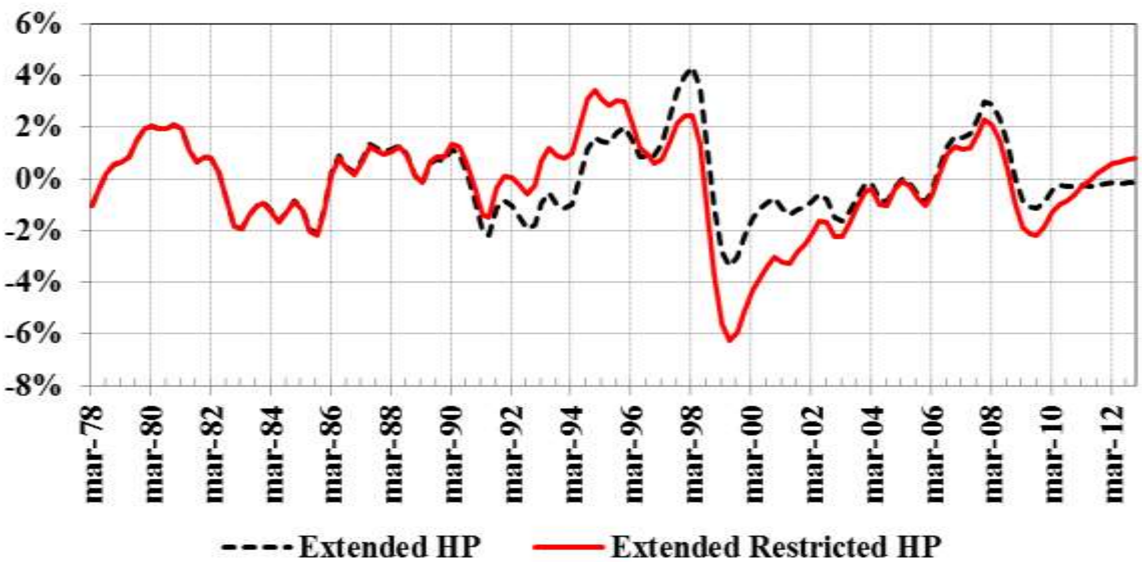
## 4. Linear Restrictions on the Extended HP Filter

The extended HP filter is meant to solve to two important drawbacks of the HP filter. On one hand, the seasonally adjusted component of the log original series still contains the noise component, which transmits to the estimated cycle causing excess noise. On the other hand, the ordinary HP filter is a single tail approximation to a symmetric two tail filter at both ends of the sample. Therefore, the HP filter estimate at both ends is biased.

In the extended filter the original series is extended with out of sample back casts and forecasts of the series in order to obtain a more approximate two sided filter at both ends of the sample. In addition, the trend-cycle component of the log extended series is filtered instead of its seasonal adjustment. The resulting filter is more resistant to the end of sample problem and excess noise than the ordinary HP filter.

Figure 2 contains the estimated output gaps obtained by the extended HP filter and the extended restricted HP filter. The extension is performed for seven years at both ends of the sample using the ARIMA model in Julio (2011a), and therefore the third, out of sample restriction applies. The figure shows data up to 2012Q4.

**Figure 2. Extended and Restricted Extended Estimates of the Colombian Output Gap**




The restriction over the cycle in 2011 has an important effect on the estimated filter. The extended filter reaches zero more slowly than the extended restricted version, and remains close to zero for an extended period of time. Moreover, the extended restricted version of the filter has a high slope at the end of the sample showing the creation of inflationary pressures for the near future.

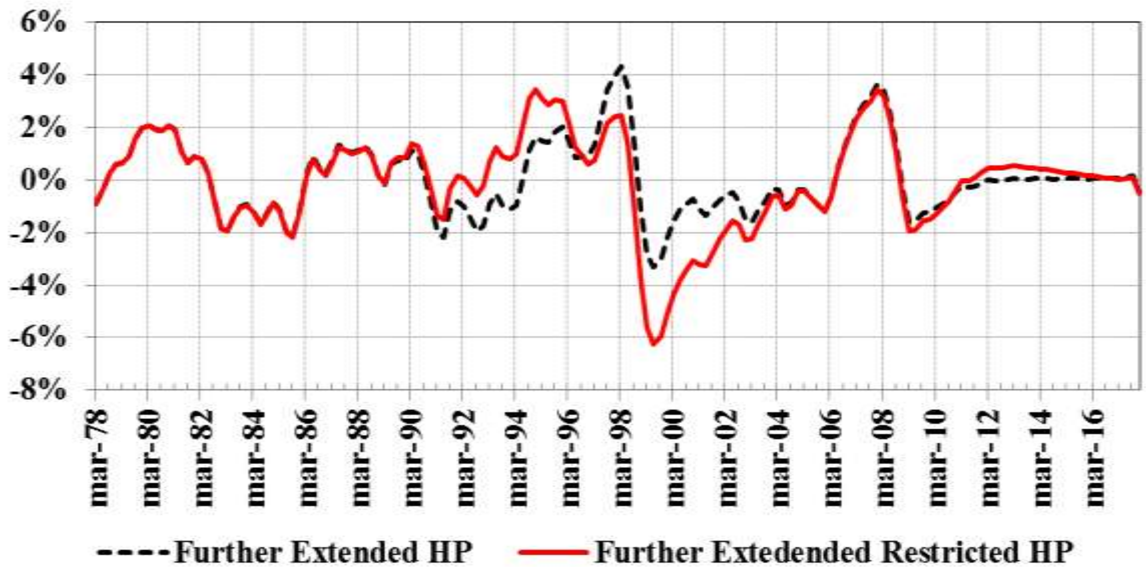
Compared to the ordinary filters the extended and extended versions are smoother.

## 5. The Further Extended HP Filter

In addition to reduce the effect of noise and the end of sample problems, the further extension of the HP filter corrects for data revisions and delays. This can be achieved by replacing the original sample with now-casts of a model for data revisions and delays, and by extending the original sample with forecasts of the same model. Out of sample back-casts can be obtained from the same model also. The trend-cycle component of the new sample is filtered through ordinary HP filtering techniques.

For this exercise we obtain in sample now-casts and seven year forecasts of the GDP data revisions model proponed by Julio (2011a) and with seven years of out of sample back-casts of the ARIMA model in Julio (2011b). The now-casts cover the last three years of the sample. Figure 3 contains the estimated gaps using further extended filtration.

**Figure 3. Further Extended Further Extended Restricted HP Filter**




Important differences in levels can be observed at the end of the sample with respect to extended filters. However, the further extended filter differ slightly form the restricted version at the very end of the sample.

## References


**Cogley, T., and Nason, J. M. (1995)** "Effects of the Hodrick–Prescott filter on trend and difference stationary time series. Implications for business cycle research" Journal of Economic Dynamics and Control 19, 253–278.

**Hodrick, E. and Prescott, E. (1997)** "Postwar U.S. Business Cycles: An Empirical Investigation", Journal of Money, Credit and Banking, Vol. 29, No. 1 (Feb., 1997), pp. 1-16.

**Julio, J. M. (2011a)** "Modeling Data Revisions", Borradores Semanales de Economía, # 641, February, BANCO DE LA REPUBLICA, http://www.banrep.gov.co/docum/ftp/borra641.pdf.

**Julio, J. M. (2011b)** "Data Revisions and the Output Gap", Borradores Semanales de Economía, # 642, February, BANCO DE LA REPUBLICA, http://www.banrep.gov.co/docum/ftp/borra642.pdf.

**Kaiser, R. and Maravall, A. (1999)** "Estimation of the business cycles: A modified Hodrick-Prescott filter", Spanish Economic Review, 1, 175-206.

**Kaiser, R. and Maravall, A. (2005)** "Combining filter design with model-based filtering (with an application to business-cycle estimation)", International Journal of Forecasting, 21, 691– 710.

**King, R.G., and Rebelo, S.T. (1993)** "Low frequency filtering and real business cycles", Journal of Economic Dynamics and Control, 17, 207–232.